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Dynamic Flexibility Method with Hybrid Shifting Frequency for Eigenvector Derivatives

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When concentrated roots exist, the precision of the dynamic flexibility method proposed previously by the authors is changed severely to be poor because a power series in the method closes in divergency. For this reason, a dynamic flexibility method with hybrid shifting frequency is developed. This new dynamic flexibility method applies a hybrid shifting frequency technique, that is, two shifting frequency values, $\Delta\omega_1$ and $\Delta\omega_2$, are put up. So a hybrid shifting frequency system is obtained. In this system $\Delta\omega_1$ is used to guarantee that the stiffness matrix of the system is always nonsingular, and $\Delta\omega_2$ is employed to accelerate the convergence of the power series contained in the present system. Thus the dynamic flexibility method with hybrid shifting frequency is suitable to the concentrated root condition. However, the eigenvector derivative of this hybrid shifting frequency system is not the same as that of the original system. To give the eigenvector derivatives of the original system, the relationship between dynamic flexibility matrices of the original and the hybrid shifting frequency system is first established. Then the eigenvector derivatives of the original system can be found from the eigenvector derivatives of the hybrid shifting frequency system. In words, this method is powerful and suitable for any (constrained and free) structures with concentrated roots.

Nomenclature

A_0, A_1	=	(n, n) matrices to be determined
$F(\lambda) \in R^{n,n}$	=	dynamic flexibility matrix
$F^{\Delta}(\lambda^*)$	=	frequency-shifted dynamic flexibility matrix
$K \in R^{n,n}$	=	real symmetric positive semidefinite stiffness matrix
$K' \in R^{n,n}$	=	derivative of stiffness matrix with respect to p_j
K^*, K^{Δ}	=	frequency shifted stiffness matrices
M	=	real symmetric positive definite mass matrix
M'	=	derivative of mass matrix with respect to p_j
p_j	=	jth design parameter
$S(\lambda) \in R^{n,n}$	=	dynamic stiffness matrix
$S^{\Delta}(\lambda^*)$	=	frequency-shifted dynamic stiffness matrix
$Z \in R^{n,m}$	=	eigenvector matrix of repeated root
$Z' \in R^{n,m}$	=	derivative matrix of eigenvector of repeated root with respect to p_j
$\Lambda' \in R^{m,m}$	=	derivative diagonal matrix of repeated root λ with respect to p_j
λ	=	repeated eigenvalue of noumenon-mode eigenvector Z
λ^*	=	frequency-shifted repeated eigenvalue of noumenon-mode eigenvector Z
$\Phi \in R^{n,n}$	=	complete eigenvector matrix including the eigenvectors Φ_k and Φ_h
$\Phi_h \in R^{n,h}$	=	higher-order eigenvector matrix excluding Φ_k
$\Phi_k \in R^{n,k}$	=	lower-order eigenvector matrix including the eigenvector Z and Φ_R
$\Phi_R \in R^{n,r}$	=	rigid-body modal matrix
$\Omega^{\Delta} \in R^{n,n}$	=	frequency-shifted eigenvalue diagonal matrix corresponding to Φ
$\Omega_h^{\Delta} \in R^{h,h}$	=	frequency-shifted higher-order eigenvalue diagonal matrix corresponding to Φ_h
$\|\cdot\|$	=	vector or matrix norm

Subscripts and Superscripts

h	=	number of higher-order eigenvectors
k	=	number of lower-order eigenvectors
m	=	number of eigenvectors of repeated root
n	=	number of degrees of freedom
r	=	number of rigid-body mode eigenvectors
T	=	transpose
$*, \Delta$	=	frequency-shifted parameter

I. Introduction

TO overcome the shortcomings of the direct methods presented in Refs. 1–4 under the case that the derivatives of many eigenvectors are computed, a dynamic flexibility (DF) method for computation of many eigenvector derivatives with repeated and unrepeated roots is first developed in Ref. 5. The primary work of the DF method is solving of matrices A_p $(p = 0, 1, \ldots)$ to be determined; however, the coefficient matrix of governing equation for solving A_p is the stiffness matrix K. Thus when there exists the number r of rigid-body motions in a structure and matrix K is singular, the present governing equation cannot be solved. For this, the number r of independent equations must be complemented into the governing equation. This way makes the coefficient matrix K become $(K + \mu M \Phi_R \Phi_R^T M)$. Obviously, this is a full matrix and breaks the band-state characteristic of original matrix K, which is unfavorable to the computational efficiency. To use the band-state characteristic of original matrix K, the author[6] adopted a shifting frequency technique to make the coefficient matrix of governing equation for solving A_p become $K^* = K - \Delta\omega M$, in which $\Delta\omega$ is the amount of shift at frequency axis. Clearly, K^* not only is always a nonsingular but also possesses the band-state characteristic of the original matrix K.

The preceding shifting frequency technique, the formal shifting frequency dynamic flexibility (FSFDF) method, not only can improve the computational efficiency but also can raise the

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convergent rate of a power series utilized in the DF method. The latter is very important because the convergent rate of the present power series will slow serially as the value of λ increases. When concentrated roots occur, this power series of λ can diverge. The FSFDF method cannot increase obviously the convergent rate of the power series, especially when concentrated roots occur because it does not change the property of original dynamic stiffness $S(\lambda) = K - \lambda M$ and dynamic flexibility $F(\lambda) = (K - \lambda M)^{-1}$. A system shifting frequency dynamic flexibility (SSFDF) method is first proposed in Ref. 7. The SSFDF method, which was originally proposed to retain the band-state of stiffness matrix K of the suspension structure, changes the property of $S(\lambda)$ and $F(\lambda)$, can accelerate the convergence of the power series, and improves the precision of calculation under the case of concentrated root. Because the coefficient matrix $K^{\Delta} = K + \Delta\omega M$ of governing equation for computing A_p in the SSFDF method is a function of frequency-shifting value $\Delta\omega$, one can only utilize one and the same value of $\Delta\omega$ for all eigenvector derivatives or uses, at most, a "divided group" shifting frequency procedure. The so-called divided group procedure means that all eigenvectors to be differentiated are divided as several groups, and then the different values of $\Delta\omega$ are adopted for a different divided group. Correspondingly, there is an "one-by-one" shifting frequency procedure. The one-by-one procedure means that the different value of $\Delta\omega$ is selected for a different eigenvector derivative. Clearly, the precision of the results obtained by the one-by-one procedure must be better than that obtained by the divided group procedure. Unfortunately, the SSFDF method[7] cannot utilize the one-by-one procedure; otherwise, the matrix K^{Δ} needs to be decomposed over and over again to make the SSFDF method lose practical value. Thus, the SSFDF cannot accelerate the convergence of the power series under concentrated root condition. For this, a noumenon-mode shifting frequency dynamic flexibility (NSFDF) method[8] is proposed. It can use an one-by-one procedure to accelerate the convergence of the power series under concentrated root condition and has better precision and efficiency. However the coefficient matrix K of governing equations in the NSFDF needs to be become as $K + \mu M \Phi_R \Phi_R^T M$, when rigid-body motion exists in a structure. This shows that the NSFDF method is not very efficient for free structure with rigid-body motion.

Based on the reason just stated, a hybrid-shifted frequency dynamic flexibility (HSFDF) method is first developed in this paper. This method is the most general shifting frequency technique. The NSFDF method is only the special format of the HSFDF method. The HSFDF method is suitable and efficient to any (constrained and free) structures with the concentrated roots. Its primary expression of calculation is an iterative formula. To improve the precision of the iteration, the accelerated iterative algorithm proposed in Ref. 9 is utilized. In addition, the present accelerated iterative algorithm[9] is simply expanded so that the best results are obtained.

Only the particular solution of eigenvector derivative is discussed in this paper, and the general solution is given by using the procedures described in Refs. 1–4.

II. Original Dynamic Flexibility Method

The principal formulas of the DF method developed by the authors are summarized as follows.[5]

Derivatives Z' of eigenvectors Z of repeated root λ are

$$Z' = F(\lambda)G \tag{1}$$

in which

$$G = MZ\Lambda' - (K' - \lambda M')Z \tag{2}$$

Because λ is the eigenvalue of eigenequation

$$(K - \omega_i M)\varphi_i = 0 \tag{3}$$

the dynamic flexibility matrix $F(\lambda) = S^{-1}(\lambda) = (K - \lambda M)^{-1}$ does not exist. For this, the following approximate perturbed dynamic flexibility formulation is utilized:

$$\bar{F}(\tilde{\lambda}) = (I - ZZ^T M)F(\tilde{\lambda}) \tag{4}$$

in which

$$F(\tilde{\lambda}) = (K - \tilde{\lambda}M)^{-1} \tag{5a}$$

$$= \Phi(\Omega - \tilde{\lambda}I)^{-1}\Phi^T \tag{5b}$$

$$= \Phi_k(\Omega_k - \tilde{\lambda}I)^{-1}\Phi_k^T + \Phi_h(\Omega_h - \tilde{\lambda}I)^{-1}\Phi_h^T \tag{5c}$$

$$= \Phi_k(\Omega_k - \tilde{\lambda}I)^{-1}\Phi_k^T + A_0 + \tilde{\lambda}A_1 + \tilde{\lambda}^2 A_2 + \cdots \tag{5d}$$

$$= \bar{\Phi}_k(\bar{\Omega}_k - \tilde{\lambda}I)^{-1}\bar{\Phi}_k^T - (\eta\lambda)^{-1}ZZ^T + A_0 + \tilde{\lambda}A_1 + \tilde{\lambda}^2 A_2 + \cdots \tag{5e}$$

Here $\Phi_k \in R^{n,k}$ is the eigenvector matrix to be differentiated in $\Phi \in R^{n,n}$ and can be expressed as $\Phi_k = [\bar{\Phi}_k, Z] = [\Phi_1, Z, \Phi_2]$, in which $\bar{\Phi}_k = [\Phi_1, \Phi_2]$. Z and λ are called as noumenon-mode eigenvector and noumenon-mode eigenvalue, respectively, and $\bar{\Phi}_k$ is a set of the lower-order additional eigenvectors. $\tilde{\lambda} = (1 + \eta)\lambda$, and η is a nondimensional small quantity. From the author's experience η can be taken as about 0.001–0.0001. The eigenvalues Ω_1 and Ω_2 corresponding, respectively, to eigenvectors Φ_1 and Φ_2 are separately little and greater than λ. For free structure Φ_1 will include rigid-body modes Φ_R. $\Phi_h \in R^{n,h}(k + h = n)$ is the higher-order eigenvector matrix. Obviously, in comparison with $F(\tilde{\lambda})$, the $\bar{F}(\tilde{\lambda})$ is more close to the following accurate value:

$$F(\lambda) = \Phi(\Omega - \lambda I)^{+}\Phi^T$$

$$= \bar{\Phi}_k(\bar{\Omega}_k - \lambda I)^{-1}\bar{\Phi}_k^T + \Phi_h(\Omega_h - \lambda I)\Phi_h^T$$

$$= \bar{\Phi}_k(\bar{\Omega}_k - \lambda I)^{-1}\bar{\Phi}_k^T + A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots \tag{6}$$

because substituting Eq. (5e) into Eq. (4) it is known that the second term as shown in right-hand side of Eq. (5e) will disappear, and this term must not be a small quantity. So the DF method's formulation for computing Z' is

$$Z' = \bar{F}(\tilde{\lambda})G \tag{7}$$

The governing equations for the matrices $A_p (p \geq 0)$ to be found in Eq. (5d) are given as[5]

$$KA_0 = I - M\Phi_k\Phi_k^T \tag{8}$$

$$KA_p = MA_{p-1}, \qquad p \geq 1 \tag{9}$$

For free structure Eqs. (8) and (9) will be replaced by the following equations:

$$\left(K + \mu M\Phi_R\Phi_R^T M\right)A_0 = I - M\Phi_k\Phi_k^T \tag{10}$$

$$\left(K + \mu M\Phi_R\Phi_R^T M\right)A_p = MA_{p-1}, \qquad p \geq 1 \tag{11}$$

where μ is a properly selected constant. Based on the experience, μ should make the elements of $\mu M\Phi_R\Phi_R^T M$ be smaller than the elements of K by $1 \sim 2$ order of magnitude.

III. HSFDF Method

It was not pointed out[10] that in the preceding DF method the matrices $A_0, A_1 \ldots$ are unchanged upon differentiating any eigenvector in Φ_k, that is, $A_p (p \geq 0)$ is not the function of eigenpair (λ, Z). This can be seen by analyzing Eqs. (8) and (9). This fact tells us that Eqs. (8) and (9) need only to be solved once regardless of the number of eigenvectors to be differentiated. This tells us also that the convergent rate of the power series at the end of Eq. (5d) will drop gradually, as the value of λ increases, because the convergent rate of the power series is nearly equivalent to that of the following geometric series:

$$1/(\omega_{k+1} - \lambda) = (1/\omega_{k+1})\left[1 + \lambda/\omega_{k+1} + (\lambda/\omega_{k+1})^2 + \cdots\right] \tag{12}$$

where ω_{k+1} is the eigenvalue corresponding to $(k+1)$th eigenvector ϕ_{k+1}, that is, ω_{k+1} is the lowest eigenvalue in higher-order mode set Φ_h. Clearly, when $\lambda \approx \omega_{k+1}$, the series as shown in Eq. (12) will diverge.

If the convergent rate of Eq. (12) under $\lambda \approx \omega_{k+1}$ condition needs to be increased obviously, one must rely on the hybrid shifting frequency techniques, which will be presented in this section.

As mentioned in the Introduction, in comparison with the SSFDF method,[7] the NSFDF method[8] is suitable to the one-by-one procedure, but as with the original DF method the coefficient matrix K of governing equations of the NSFDF method is singular under free structure condition so that the coefficient matrix must become a full matrix $(K + \mu M \Phi_R \Phi_R^T M)$. Now we want to find a shifting frequency technique that can retain the band-state characteristic of matrix K and is also suitable to the one-by-one procedure. After analyzing the respective merits of FSFDF, SSFDF, and NSFDF methods, a HSFDF method is first developed in this paper.

The HSFDF method takes the frequency-shifted dynamic flexibility matrix as follows:

$$F^{\Delta}(\lambda^*) = (K^{\Delta} - \lambda^* M)^{-1} \tag{13a}$$

$$= \Phi(\Omega^{\Delta} - \lambda^* I)^{-1}\Phi^T \tag{13b}$$

Here $\Omega^{\Delta} = \Omega + \Delta\omega_1 I$ are the eigenvalues of the following eigenequation:

$$\left(K^{\Delta} - \omega_i^{\Delta} M\right)\varphi_i = 0 \tag{14}$$

in which $K^{\Delta} = K + \Delta\omega_1 M$ and $\omega_i^{\Delta} = \omega_i + \Delta\omega_1$; λ^* is a m-multiple eigenvalue of the next eigenequation:

$$\left(K^* - \omega_i^* M\right)\varphi_i = 0 \tag{15}$$

in which $K^* = K - \Delta\omega_2 M$, $\omega_i^* = \omega_i - \Delta\omega_2$, and $\lambda^* = \lambda - \Delta\omega_2$. Assume $\Phi = [\Phi_0, Z]$ and $\Omega^{\Delta} = \mathrm{diag}[\Omega_0^{\Delta}, \lambda^{\Delta} I]$; here $\lambda^{\Delta} = \lambda + \Delta\omega_1$ is a m-multiple eigenvalue of Eq. (14). Thus, Eq. (13b) can be expanded as

$$F^{\Delta}(\lambda^*) = \Phi_0\left(\Omega_0^{\Delta} - \lambda^* I\right)^{-1}\Phi_0^T + \Delta\bar{\bar{\omega}}^{-1} Z Z^T \tag{16a}$$

$$= \bar{\Phi}_k\left(\bar{\Omega}_k^{\Delta} - \lambda^* I\right)^{-1}\bar{\Phi}_k^T + \Phi_h\left(\Omega_h^{\Delta} - \lambda^* I\right)^{-1}\Phi_h^T + \Delta\bar{\bar{\omega}}^{-1} Z Z^T \tag{16b}$$

$$= \bar{\Phi}_k\left(\bar{\Omega}_k^{\Delta} - \lambda^* I\right)^{-1}\bar{\Phi}_k^T + \Delta\bar{\bar{\omega}}^{-1} Z Z^T + A_0 + \lambda^* A_1 + \lambda^{*2} A_2 + \cdots \tag{16c}$$

$$= \Phi_k\left(\Omega_k^{\Delta} - \lambda^* I\right)^{-1}\Phi_k^T + A_0 + \lambda^* A_1 + \lambda^{*2} A_2 + \cdots \tag{16d}$$

where $\Delta\bar{\bar{\omega}} = \Delta\omega_1 + \Delta\omega_2$, $\Omega_k^{\Delta} = \Omega_k + \Delta\omega_1 I$, and $\bar{\Omega}_k^{\Delta} = \bar{\Omega}_k + \Delta\omega_1 I$, on the analogy of this.

To solve matrices $A_p (p \geq 0)$ to be determined, Eq. (16d) can be rewritten as

$$F^{\Delta}(\lambda^*) = -\sum_i \frac{\varphi_{1i}\varphi_{1i}^T}{\lambda^* - \omega_{1i}^{\Delta}} + \sum_j \frac{\varphi_{2j}\varphi_{2j}^T}{\omega_{2j}^{\Delta} - \lambda^*} + A_0 + \lambda^* A_1 + \lambda^{*2} A_2 + \cdots = -\sum_i \frac{\varphi_{1i}\varphi_{1i}^T}{\lambda^*}\left[1 + \frac{\omega_{1i}^{\Delta}}{\lambda^*} + \left(\frac{\omega_{1i}^{\Delta}}{\lambda^*}\right)^2 + \cdots\right] + \sum_j \frac{\varphi_{2j}\varphi_{2j}^T}{\omega_{2j}^{\Delta}}\left[1 + \frac{\lambda^*}{\omega_{2j}^{\Delta}} + \left(\frac{\lambda^*}{\omega_{2j}^{\Delta}}\right)^2 + \cdots\right] + A_0 + \lambda^* A_1 + \lambda^{*2} A_2 + \cdots$$

$$= -\lambda^{*-1}\Phi_1\Phi_1^T - \lambda^{*-2}\Phi_1\Omega_1^{\Delta}\Phi_1^T - \lambda^{*-3}\Phi_1\Omega_1^{\Delta 2}\Phi_1^T - \cdots + \Phi_2\Omega_2^{\Delta -1}\Phi_2^T + \lambda^*\Phi_2\Omega_2^{\Delta -2}\Phi_2^T + \lambda^{*2}\Phi_2\Omega_2^{\Delta -3}\Phi_2^T + \cdots + A_0 + \lambda^* A_1 + \lambda^{*2} A_2 + \cdots \tag{17}$$

Here the definition of Φ_1 and Φ_2 is slightly not the same as the early one, and $\Phi_k = [\Phi_1, \Phi_2]$, that is to say, the noumenon-mode eigenvector Z is contained in Φ_1 or Φ_2. Their corresponding eigenvalues $\Omega_1^{\Delta} = \Omega_1 + \Delta\omega_1 I$ and $\Omega_2^{\Delta} = \Omega_2 + \Delta\omega_1 I$ are smaller and greater than λ^*, respectively. According to the procedure described in Ref. 5, substituting Eq. (17) into the relationship $(K^{\Delta} - \lambda^* M)F^{\Delta}(\lambda^*) = I$ gets the governing equations

$$K^{\Delta} A_0 = I - M\Phi_k\Phi_k^T \tag{18}$$

$$K^{\Delta} A_p = M A_{p-1}, \qquad p \geq 1 \tag{19}$$

After getting $F^{\Delta}(\lambda^*)$ from Eq. (16d), a frequency shifted solution $Z^{\Delta'}$ of eigenvector derivative in the HSFDF method is given as

$$Z^{\Delta'} = F^{\Delta}(\lambda^*) G \tag{20}$$

Note that $Z^{\Delta'}$ is not equal to the following real solution:

$$Z' = F(\lambda) G \approx \bar{F}(\tilde{\lambda}) G \tag{21}$$

To derive Z' from $Z^{\Delta'}$, the relationship between $\bar{F}(\tilde{\lambda})$ and $F^{\Delta}(\lambda^*)$ must be found. From Eq. (5b) one knows

$$F(\tilde{\lambda}) = \Phi_0(\Omega_0 - \tilde{\lambda} I)^{-1}\Phi_0^T - (\eta\lambda)^{-1} Z Z^T \tag{22}$$

Embedding Eq. (22) into Eq. (4) and using the mass-orthogonality condition makes

$$\bar{F}(\tilde{\lambda}) = \Phi_0(\Omega_0 - \tilde{\lambda} I)^{-1}\Phi_0^T \tag{23a}$$

$$= \bar{\Phi}_k(\bar{\Omega}_k - \tilde{\lambda} I)^{-1}\bar{\Phi}_k^T + \Phi_h(\Omega_h - \tilde{\lambda} I)^{-1}\Phi_h^T \tag{23b}$$

Thus, using Eqs. (16a) and (23a) gives

$$F^{\Delta}(\lambda^*) - \bar{F}(\tilde{\lambda}) = \Phi_0\left[\left(\Omega_0^{\Delta} - \lambda^* I\right)^{-1} - (\Omega_0 - \tilde{\lambda} I)^{-1}\right]\Phi_0^T + \Delta\bar{\bar{\omega}}^{-1} Z Z^T \tag{24}$$

Analyzing the ith diagonal element of the diagonal matrix in Eq. (24) has

$$\frac{1}{\omega_i^{\Delta} - \lambda^*} - \frac{1}{\omega_i - \tilde{\lambda}} = -\frac{\Delta\bar{\bar{\omega}} + \eta\lambda}{\left(\omega_i^{\Delta} - \lambda^*\right)(\omega_i - \tilde{\lambda})} \tag{25}$$

Equation (24) can rewritten by using Eq. (25) as

$$F^{\Delta}(\lambda^*) - \bar{F}(\tilde{\lambda}) = -(\Delta\bar{\bar{\omega}} + \eta\lambda)\Phi_0\left(\Omega_0^{\Delta} - \lambda^* I\right)^{-1}(\Omega_0 - \tilde{\lambda} I)^{-1}\Phi_0^T + \Delta\bar{\bar{\omega}}^{-1} Z Z^T = -(\Delta\bar{\bar{\omega}} + \eta\lambda)\bar{\Phi}_k\left(\bar{\Omega}_k^{\Delta} - \lambda^* I\right)^{-1}(\bar{\Omega}_k - \tilde{\lambda} I)^{-1}\bar{\Phi}_k^T - (\Delta\bar{\bar{\omega}} + \eta\lambda)\Phi_h\left(\Omega_h^{\Delta} - \lambda^* I\right)^{-1}(\Omega_h - \tilde{\lambda} I)^{-1}\Phi_h^T + \Delta\bar{\bar{\omega}}^{-1} Z Z^T \tag{26}$$

Using the relationship $\Phi_k^T M \Phi_h = 0$, $\Phi_h^T M \Phi_h = I$, as well as Eqs. (16b) and (23b), etc., the matrix in the second term on the right-hand side of Eq. (26) can be rewritten as

$$\Phi_h\left(\Omega_h^{\Delta} - \lambda^* I\right)^{-1}\Phi_h^T M \Phi_h(\Omega_h - \tilde{\lambda} I)^{-1}\Phi_h^T = \left[F^{\Delta}(\lambda^*) - \Phi_k\left(\Omega_k^{\Delta} - \lambda^* I\right)^{-1}\Phi_k^T\right] M \Phi_h(\Omega_h - \tilde{\lambda} I)^{-1}\Phi_h^T = F^{\Delta}(\lambda^*) M\left[\bar{F}(\tilde{\lambda}) - \bar{\Phi}_k(\bar{\Omega}_k - \tilde{\lambda} I)^{-1}\bar{\Phi}_k^T\right] = F^{\Delta}(\lambda^*) M \bar{F}(\tilde{\lambda}) - \bar{\Phi}_k\left(\bar{\Omega}_k^{\Delta} - \lambda^* I\right)^{-1}(\bar{\Omega}_k - \tilde{\lambda} I)^{-1}\bar{\Phi}_k^T \tag{27}$$

Substituting Eq. (27) into Eq. (26) obtains

$$F^{\Delta}(\lambda^*) - \bar{F}(\tilde{\lambda}) = -(\Delta\bar{\bar{\omega}} + \eta\lambda) F^{\Delta}(\lambda^*) M \bar{F}(\tilde{\lambda}) + \Delta\bar{\bar{\omega}}^{-1} Z Z^T \tag{28}$$

Postmultiplying both sides of Eq. (28) by matrix G gives

$$[I - (\Delta\bar{\bar{\omega}} + \eta\lambda) F^{\Delta}(\lambda^*) M] Z' = Z^{\Delta'} - \Delta\bar{\bar{\omega}}^{-1} Z Z^T G \tag{29}$$

Equation (29) can yield the exact solution of Z' when not considering the approximation of $F^{\Delta}(\lambda^*)$ and η, but this equation is unsuitable to the calculation of many eigenvector derivatives because the coefficient matrix of Eq. (29) is a function of λ and $\Delta\bar{\bar{\omega}}$. For this reason, Eq. (29) is rewritten as an iterative formula of the HSFDF method:

$$Z'_{(k)} = Z^{\Delta'} - \Delta\bar{\bar{\omega}}^{-1} ZZ^T G + (\Delta\bar{\bar{\omega}} + \eta\lambda) F^{\Delta}(\lambda^*) M Z'_{(k-1)} \qquad k \geq 1 \quad (30)$$

The convergent rate of the power series in Eq. (16) is nearly equivalent to that of the following geometric series:

$$1/\left(\omega^{\Delta}_{k+1} - \lambda^*\right) = 1/\omega^{\Delta}_{k+1}\left[1 + \lambda^*/\omega^{\Delta}_{k+1} + \left(\lambda^*/\omega^{\Delta}_{k+1}\right)^2 + \cdots\right] \quad (31)$$

Because $\omega^{\Delta}_{k+1} = \omega_{k+1} + \Delta\omega_1$ and $\lambda^* = \lambda - \Delta\omega_2$, even if under $\lambda \approx \omega_{k+1}$ condition, the ratio $\lambda^*/\omega^{\Delta}_{k+1}$ can be ensured to be very small provided both $\Delta\omega_1$ and $\Delta\omega_2$ are selected properly.

Finally, it is pointed that the superiority of the HSFDF method is that two shifting frequency quantities, $\Delta\omega_1$ and $\Delta\omega_2$, are put up. For all eigenvector derivatives one needs only to take one and the same value of $\Delta\omega_1$ to ensure the unsingularity of matrix K^{Δ} and to make certain K^{Δ} needs only to be decomposed once. Also K^{Δ} possesses the band-state characteristic of K. Oppositely, for different eigenvector derivatives one needs to select a different value of $\Delta\omega_2$ to guarantee λ^* always close to zero. In addition, using Eqs. (18) and (19) the matrices $A_p\,(p \geq 0)$ need only to be computed once. These ensure the efficiency of the HSFDF method.

IV. Solution of Iterative Procedure

For the convenience of description, Eq. (30) is called a basic iterative formula and is rewritten as

$$Z'_{(k)} = B + XZ'_{(k-1)}, \qquad k \geq 1 \quad (32)$$

in which

$$B = Z^{\Delta'} - \Delta\bar{\bar{\omega}}^{-1} ZZ^T G \quad (33)$$

$$X = (\Delta\bar{\bar{\omega}} + \eta\lambda) F^{\Delta}(\lambda^*) M \quad (34)$$

If $Z'_{(0)}$ is taken as zero, then the preceding procedure means that the real initial vector can be considered to be $Z'_{(0)} = B$. Thus, the iterative process of Eq. (32) can be described, in detail, as

$$\left.\begin{aligned} Z'_{(0)} &= B \\ Z'_{(1)} &= B + XB = (I + X)B \\ Z'_{(2)} &= B + X(I + X)B = (I + X + X^2)B \\ &\vdots \\ Z'_{(k)} &= (I + X + X^2 + \cdots + X^k)B \end{aligned}\right\} \quad (35)$$

If the iterative process of Eq. (35) is required to be convergent, the norm inequality $\|X\| < 1$ must be guaranteed. Under the prerequisite of $\|X\| < 1$, the following limitation expression exists:

$$\lim_{k \to \infty} Z'_{(k)} = [I - X]^{-1} B = Z' \quad (36)$$

Substituting Eqs. (33) and (34) into Eq. (36) obtains

$$Z' = [I - (\Delta\bar{\bar{\omega}} + \eta\lambda) F^{\Delta}(\lambda^*) M]^{-1} (Z^{\Delta'} - \Delta\bar{\bar{\omega}}^{-1} ZZ^T G) \quad (37)$$

From Eq. (29) one knows that Eq. (37) is just the accurate solution of the HSFDF method. This shows that under the prerequisites of $\|X\| < 1$ and $Z'_{(0)} = 0$ or B, the iterative formula (30) can converge to the exact solution.

To accelerate the convergence of iterative process shown in Eq. (32), we use the accelerated iterative algorithm 2 presented in Ref. 9. For convenience of description, let $Z' = [z'_1, z'_2, \ldots, z'_m]$ and $B = [b_1, b_2, \ldots, b_m]$, and z' and b are considered, respectively, as certain column vectors with the same number of columns in matrices Z' and B. The algorithm 2 in Ref. 9 can be expressed as follows.

For the calculation of a certain eigenvector derivative z', the initial parameter is

$$\Xi_0 = \left[\xi_0^{(1)}, \xi_0^{(2)}\right]^T = (0, 0)^T \quad (38a)$$

For $k = 1, 2, \ldots$

$$u_k^{(1)} = b + \xi_{k-1}^{(1)}\left[u_{k-1}^{(2)} - b\right] + \xi_{k-1}^{(2)}\left[u_{k-1}^{(3)} - b\right] \quad (38b)$$

$$u_k^{(2)} = b + Xu_k^{(1)} \quad (38c)$$

If $\|u_k^{(2)} - u_k^{(1)}\|/\|u_k^{(2)}\| < \sigma$, stop, and let $z' = u_k^{(2)}$; otherwise,

$$u_k^{(3)} = b + Xu_k^{(2)} \quad (38d)$$

$$\beta_k = \left\{\left[u_k^{(1)} - u_k^{(2)} + b\right], \left[u_k^{(2)} - u_k^{(3)} + b\right]\right\} \quad (38e)$$

$$\rho_k = \left[u_k^{(1)}, u_k^{(2)}\right] \quad (38f)$$

$$\Xi_k = \left[\rho_k^T M \beta_k\right]^{-1} \rho_k^T Mb = \left[\xi_k^{(1)}, \xi_k^{(2)}\right]^T \quad (38g)$$

Let $k := k + 1$, then return to the computation of Eqs. (38b) to (38c), where σ is a given small number, $\sigma > 0$.

A simply expanded and more sound alternative, as compared with the preceding accelerated iterative procedure, is presented in the Appendix.

V. Proof of Method

The shifting frequency dynamic flexibility method proposed in this paper first builds up a frequency-shifted system and solves eigenvector derivatives $Z^{\Delta'}$ of this system, and then eigenvector derivatives Z' of real system are computed by using $Z^{\Delta'}$ in terms of the established relationship between frequency-shifted system and the original real system. Now we ask whether the results obtained by Eq. (30) are the eigenvector derivatives of original real system. To answer this question, we eliminate the iterative subscripts shown in Eq. (30) such that there is

$$Z' = Z^{\Delta'} - \Delta\bar{\bar{\omega}}^{-1} ZZ^T G + (\Delta\bar{\bar{\omega}} + \eta\lambda) F^{\Delta}(\lambda^*) M Z' \quad (39)$$

Inserting Eq. (20) into Eq. (39), premultiplying both sides of Eq. (39) by $[K^{\Delta} - \lambda^* M]$, and using relationship $[K^{\Delta} - \lambda^* M] F^{\Delta}(\lambda^*) = I$ results in

$$[K^{\Delta} - \lambda^* M] Z' = G - \Delta\bar{\bar{\omega}}^{-1} [K^{\Delta} - \lambda^* M] ZZ^T G + (\Delta\bar{\bar{\omega}} + \eta\lambda) MZ' \quad (40)$$

Next, introducing relations $\lambda^* = \lambda - \Delta\omega_2$, $\tilde{\lambda} = (1 + \eta)\lambda$, $[K - \lambda M] Z = 0$, and Eq. (5a) into Eq. (40), one finds

$$Z' = F(\tilde{\lambda})(I - MZZ^T) G = \bar{F}(\tilde{\lambda}) G \quad (41)$$

Equations (4), (7), and (41) tell us that the result given by the HSFDF method is just the eigenvector derivative of original real system.

Another proof of the method is that premultiplying Eq. (29) by positive definite matrix M, using Eq. (13b), and introducing the relationship[5,10] $M = M\Phi\Phi^T M$ can again derive from $Z' = \bar{F}(\tilde{\lambda}) G$. This shows again that the HSFDF method is, in theory, completely correct.

VI. Numerical Example

The numerical example used in this paper is similar to one in Ref. 3, except that one beam element is added and the fixed end is released. Thus, this suspension beam has 16 degrees of freedom. Each beam element has the same square cross section. The derivative calculation is made with respect to the design parameter p_j. Here p_j is chosen to be the z-axis area moment I_z of the beam element at the top end. The finite element assembly matrices K and M of the suspension beam as well as their derivative matrices K' and M' are obtained easily by Ref. 3. The present example possesses eight double roots designated from λ_1 to λ_8 in which $\lambda_1 = \lambda_2 = 0$. The eigenvectors corresponding to λ_1 and λ_2 are the rigid-body modes

Table 1 Percentage error of eigenvector derivative of both DF and HSFDF methods

Mode	Order of series in $\bar{F}(\tilde{\lambda})$ and $F^{\Delta}(\lambda^*)$	DF method	HSFDF method				
			$\Delta\omega_2$	Accurate solution	Basic iteration	Accelerated iteration	Expanded accelerated iteration
6	1st	0.00171	14.7948	$0.215E-9$	$0.552E-4(6)$	$0.281E-4(2)$	$0.149E-6(2)$
	2nd	$0.146E-4$	(14.7948)	$0.215E-9$	$0.552E-4(6)$	$0.281E-4(2)$	$0.149E-6(2)$
	3rd	$0.137E-6$	[14.7948]	$0.215E-9$	$0.552E-4(6)$	$0.281E-4(2)$	$0.149E-6(2)$
	4th	$0.716E-9$	<14.7948>	$0.215E-9$	$0.552E-4(6)$	$0.281E-4(2)$	$0.149E-6(2)$
8	1st	0.172	112.9641	$0.505E-10$	0.07810(20)	$0.225E-3(13)$	$0.229E-5(4)$
	2nd	0.0112	(37.654)	$0.505E-10$	0.00473(20)	$0.225E-3(13)$	$0.187E-5(4)$
	3rd	$0.720E-3$	[112.9641]	$0.505E-10$	0.00330(20)	$0.225E-3(13)$	$0.187E-5(4)$
	4th	$0.465E-4$	<112.9641>	$0.505E-10$	0.00330(20)	$0.217E-3(13)$	$0.187E-5(4)$
10	1st	13.947	543.4680	$0.119E-9$	6.987(20)	2.020(7)	$0.867E-3(8)$
	2nd	4.332	(181.156)	$0.118E-9$	1.382(20)	**0.209(6)**	**$0.467E-3(8)$**
	3rd	**1.345**	[362.0]	$0.118E-9$	0.218(20)	0.0212(8)	$0.467E-3(8)$
	4th	0.418	<543.4680>	$0.118E-9$	0.0845(20)	0.00236(6)	$0.467E-3(8)$

Table 2 Eigenvector derivatives of the fifth double root

Exact value		DF method, $\mu = 10$		HSFDF method: Accelerated iteration, $\Delta\omega_2 = 362$		HSFDF method: Expanded accelerated iteration, $\Delta\omega_2 = 543.468$	
$z'_{5,1}$	$z'_{5,2}$	$z'_{5,1}$	$z'_{5,2}$	$z'_{5,1}$	$z'_{5,2}$	$z'_{5,1}$	$z'_{5,2}$
0.0	$0.31788E-4$	0.0	$0.14959E-3$	0.0	$0.50088E-4$	0.0	$0.31738E-4$
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0
0.0	$0.52372E-1$	0.0	$0.51634E-1$	0.0	$0.52257E-1$	0.0	$0.52372E-1$
0.0	$-0.13967E-1$	0.0	$-0.13917E-1$	0.0	$-0.13959E-1$	0.0	$-0.13967E-1$
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0
0.0	$-0.42448E-1$	0.0	$-0.42027E-1$	0.0	$-0.42382E-1$	0.0	$-0.42448E-1$
0.0	$0.11770E-1$	0.0	$0.11721E-1$	0.0	$0.11763E-1$	0.0	$0.11771E-1$
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0
0.0	$-0.43599E-1$	0.0	$-0.43178E-1$	0.0	$-0.43533E-1$	0.0	$-0.43599E-1$
0.0	$0.17832E-2$	0.0	$0.16655E-2$	0.0	$0.17650E-2$	0.0	$0.17834E-2$
0.0	0	0.0	0	0.0	0	0.0	0
0.0	0	0.0	0	0.0	0	0.0	0
0.0	$0.36901E-1$	0.0	$0.36164E-1$	0.0	$0.36786E-1$	0.0	$0.36901E-1$
$z'\%$	——	0.0	1.345	0.0	0.209	0.0	$0.467E-3$

$\Phi_R \in R^{16,4}$. Here taking the first 10 lower-order modes forms the eigenvector matrix Φ_k, then the derivatives of six eigenvectors of $\lambda_3 \sim \lambda_5$ in Φ_k are computed by using Eqs. (7), (29), (30), (38), and (A1). Here $\lambda_3 = 14.79487$, $\lambda_4 = 112.96418$, and $\lambda_5 = 543.46809$, as well as $\lambda'_{3,1} = \lambda'_{4,1} = \lambda'_{5,1} = 0$, $\lambda'_{3,2} = 2.0651$, $\lambda'_{4,2} = 39.847$, and $\lambda'_{5,2} = 202.44$. The computed results are listed in Tables 1 and 2. Note that $\Delta\omega_1 = 1.0$ for all iterations of HSFDF method, and $\sigma = 10^{-5}$ and $\eta = 0.0001$ are adopted in the calculations.

To understand in detail the characteristics of various dynamic flexibility methods, the precision analysis (that is, percentage error) of various methods with different approximation of the power series as shown in $\bar{F}(\tilde{\lambda})$ and $F^{\Delta}(\lambda^*)$ is given in Table 1. The "1st" and "2nd," etc., mentioned in Table 1 represent the order of power terms of $\lambda^{*0} \sim \lambda^{*1}$ (or $\tilde{\lambda}^0 \sim \tilde{\lambda}^1$), $\lambda^{*0} \sim \lambda^{*2}$ ($\tilde{\lambda}^0 \sim \tilde{\lambda}^2$), etc., as shown in Eqs. (5d) and (16d). In Table 1 the values without brackets and the values with parentheses and square and angle brackets shown in the $\Delta\omega_2$ term indicate the shifting frequency values used in the accurate solution, the basic iterative, the accelerated, and the expanded accelerated iterative solutions, respectively; and integral numbers with parentheses shown in the iterative solution term stand for the number of iteration. The percentage change of an eigenvector derivative z' with respect to the exact solution z'_e is defined in Eq. (42):

$$z'\% = \left(\frac{\| z'_e - z' \|}{\| z'_e \|} \right)^{\frac{1}{2}} \% \tag{42}$$

Because the first eigenvector derivative of each double root is zero vector, the corresponding percentage error is zero; therefore, the percentage errors of them are not listed in Table 1. As a result, modes 6, 8, and 10 in Table 1, respectively, denote the second eigenvector of the third, fourth, and fifth double root. In addition, because the difference between the lower-order eigenvector derivatives obtained by DF and HSFDF methods is not obvious as a result of nonexistence of concentrated roots, only the derivatives of the highest-order eigenvector in Φ_k corresponding to the percentage error data in boldface shown in Table 1 are listed in Table 2.

1) The accurate and iterative solutions yielded, respectively, by Eqs. (29) and (30) plus Eq. (A1) show that the idea of the HSFDF method is completely correct because Eqs. (29) and (30) plus Eq. (A1) allow $\lambda^* \approx 0$ in terms of the one-by-one procedure, so that the $F^{\Delta}(\lambda^*)$ converges rapidly. Thus, when using Eqs. (29) and (30) plus Eq. (A1) and taking the first-order (even zero-order) approximation of $F^{\Delta}(\lambda^*)$ can give very exact results (see Table 1). Unfortunately, Eq. (29) cannot be used in the calculation of many eigenvector derivatives because it leads to the loss of merits of dynamic flexibility method.[5] Thus, Eq. (30) plus the expanded accelerated iterative algorithm shown in the Appendix will be most useful in practice.

2) The basic and accelerated iterative solutions do not allow $\lambda^* \approx 0$, that is, they do not allow $\Delta\omega_2$ to approach to λ because the convergence of Eq. (30) is influenced severely by the magnitude of $\Delta\omega_2$ (that is, $\Delta\bar{\omega}$). Numerical experience (see Table 1) tells us that there is the requirement $\Delta\omega_2^{(i)} \approx \lambda_i/3$ for the ith eigenvalue λ_i for the basic iterative procedure of Eq. (30). However for the accelerated iterative procedure of Eq. (38), only $\Delta\omega_2^{(i)} \approx 2\lambda_i/3$ is required. This shows that the effect of the accelerated iteration is obviously better than that of the basic iteration.

3) At present the precision of accelerated iterative formula (38) can also satisfy the needs of engineering problems, provided the requirement $\Delta\omega_2 \approx 2\lambda/3$ is met and third-order (at most fourth-order)

approximation of $F^{\Delta}(\lambda^*)$ is used. Note that $\Delta\omega_2 \approx 2\lambda/3$ can make the $\lambda/\omega_{k+1} \approx 1$ condition in Eq. (12) change the $\lambda^*/\omega_{k+1}^{\Delta} \approx \frac{1}{3}$ condition in Eq. (31); this can mostly separate the concentrated roots.

4) For the HSFDF method the unsingularity of matrix K^{Δ} is always guaranteed by using parameter ω_1. Under the prerequisite of K^{Δ} being nonsingular, $\Delta\omega_1$ should be chosen as small as probable.

5) A shortcoming of this example is that there are no concentrated roots. However through the comparison of derivative of highest-order eigenvector in Φ_k, one can understand that accuracy of the HSFDF method must be obviously better than that of the existing DF method under the concentrated root condition.

6) Letting $\Delta\omega_1 = 0$ and $\Delta\omega_2 = \Delta\omega$ in the formulas of the HSFDF method, the formulas of the NSFDF method presented in Ref. 8 can be found.

7) Three noniterative formulas corresponding to Eq. (30) can be derived out. These noniterative formulas are similar to those of the simplified dynamic flexibility (SDF)[10] and the higher precision dynamic flexibility (HPDF)[11] methods. Two formulas in the noniterative formulas just stated are just the noniterative formulas of the SDF and HPDF methods with hybrid shifting frequency, respectively. However the iterative formula of the SDF and the HPDF methods with hybrid shifting frequency still is Eq. (30), in which only $F^{\Delta}(\lambda^*)$ corresponding to the SDF and HPDF methods with hybrid shifting frequency are given, respectively, by the following expressions:

$$F^{\Delta}(\lambda^*) = \Phi_k\left(\Omega_k^{\Delta} - \lambda^* I\right)^{-1}\Phi_k^T + \sum_{p=0}^{s}\lambda^{*p}\left(R_h^{\Delta}M\right)^p R_h^{\Delta} \tag{43}$$

$$F^{\Delta}(\lambda^*) = \Phi_k\left(\Omega_k^{\Delta} - \lambda^* I\right)^{-1}\Phi_k^T + \sum_{p=0}^{s}\lambda^{*p}\Psi_h A_p \Psi_h^T \tag{44a}$$

or

$$F^{\Delta}(\lambda^*) = \Phi_k\left(\Omega_k^{\Delta} - \lambda^* I\right)^{-1}\Phi_k^T - \sum_{p=1}^{s}\lambda^{*-p}\Psi_h A_{-p} \Psi_h^T \tag{44b}$$

in which solving of A_p or A_{-p} is described in Ref. 10. However, the κ_h in the equations for solving A_p or A_{-p} shall be κ_h^{Δ}.

VII. Conclusions

1) For the constrained structure with concentrated roots and any constrained structural higher-order eigenvector derivative, adopting the NSFDF method is most suitable because it can obtain very good eigenvector derivatives via the one-by-one procedure and does not need also to attempt to retain the band-state characteristic of matrix K through the shifting frequency technique.

2) For the free structure with concentrated roots and any free structural higher-order eigenvector derivative, using the HSFDF method is most reasonable because it can retain the band state of matrix K via a unique value of $\Delta\omega_1$ and can also realize the one-by-one procedure through the different values of $\Delta\omega_2$.

3) For the constrained structure without concentrated roots and any constrained structural lower-order eigenvector derivative, one can use the original DF method. For the free structure without concentrated roots and any free structural lower-order eigenvector derivative, adopting the simple FSFDF method can give satisfactory eigenvector derivatives and can retain the band-state of matrix K.

4) The expanded accelerated iterative procedure will be most useful because it can make $\lambda^* \approx 0$ and the concentrated roots be separated enough.

5) In practice the NSFDF method is a special case of the HSFDF method because the HSFDF method is degenerated to the NSFDF method after letting $\Delta\omega_1 = 0$ and $\Delta\omega_2 = \Delta\omega$. This shows that the HSFDF method is the general shifting frequency technique.

Appendix: Expanded Accelerated Iterative Algorithm

The algorithm 2 proposed in Ref. 9 is simply expanded here by introducing three variable parameters. The expanded accelerated iterative (EAI) algorithm is only referable because its rationality is not demonstrated strictly in mathematics. However the numerical results show that it is a very efficient algorithm. The EAI algorithm is presented as follows.

For the certain eigenvector derivative z' the initial parameter is

$$\Xi_0 = \left[\xi_0^{(1)}, \xi_0^{(2)}, \xi_0^{(3)}\right]^T = (0, 0, 0)^T \tag{A1a}$$

For $k = 1, 2, \ldots$

$$u_k^{(1)} = b + \xi_{k-1}^{(1)}\left[u_{k-1}^{(2)} - b\right] + \xi_{k-1}^{(2)}\left[u_{k-1}^{(3)} - b\right] + \xi_{k-1}^{(3)}\left[u_{k-1}^{(4)} - b\right] \tag{A1b}$$

$$u_k^{(2)} = b + Xu_k^{(1)} \tag{A1c}$$

If $\|u_k^{(2)} - u_k^{(1)}\|/\|u_k^{(2)}\| < \sigma$, stop, and let $z' = u_k^{(2)}$; otherwise,

$$u_k^{(3)} = b + Xu_k^{(2)} \tag{A1d}$$

$$u_k^{(4)} = b + Xu_k^{(3)} \tag{A1e}$$

$$\beta_k = \left\{\left[u_k^{(1)} - u_k^{(2)} + b\right], \left[u_k^{(2)} - u_k^{(3)} + b\right], \left[u_k^{(3)} - u_k^{(4)} + b\right]\right\} \tag{A1f}$$

$$\rho_k = \left[u_k^{(1)}, u_k^{(2)}, u_k^{(3)}\right] \tag{A1g}$$

$$\Xi_k = \left[\rho_k^T M\beta_k\right]^{-1}\rho_k^T Mb = \left[\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}\right]^T \tag{A1h}$$

Let $k := k + 1$, then return to the calculation of Eqs. (A1b) and (A1c).

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